

Axiomatization of Finite Algebras

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Abstract. We show that the set of all formulas in n variables valid in a finite class \mathbf{A} of finite algebras is always a regular tree language, and compute a finite axiom set for \mathbf{A} . We give a rational reconstruction of Barzdins’ *liquid flow algorithm* [BB91]. We show a sufficient condition for the existence of a class \mathbf{A} of *prototype algebras* for a given theory Θ . Such a set allows us to prove $\Theta \models \varphi$ simply by testing whether φ holds in \mathbf{A} .

1 Introduction

Abstraction is a key issue in artificial intelligence. In the setting of mathematical logic and model theory, it is concerned with the relation between *concrete* algebras and *abstract* statements about them in a formal language. For purely equational theories, the well-known construction of an initial model (e.g. [DJ90]) allows one to compute a kind of prototypical algebra for a given theory. In the other direction (i.e. from concrete algebras to theories), however, no computable procedures are yet known. While it is trivial to *check* whether a given formula is valid in a given finite algebra, it is not clear how to *find* a finite description of all valid formulas.

In 1991, Barzdin and Barzdin [BB91] proposed their *liquid-flow algorithm* which takes an incompletely given finite algebra and acquires *hypotheses* about what are probable axioms. We give a rational reconstruction of this work that is based on well-known algorithms on regular tree grammars. We give a correspondence between Barzdins’ notions and grammar notions, showing that the liquid-flow algorithm in fact amounts to a combination of classical grammar algorithms (Thm. 10).

The correspondence leads to synergies in both directions: Barzdins’ approach could be extended somewhat, and a classical algorithm seems to be improvable in its time complexity using the liquid-flow technique.

Next, we focus on a completely given algebra and show how to compute finite descriptions of the set of all variable-bounded formulas valid in it. This set is described by a grammar (Thm. 11) and by an axiom set (Thm. 13).

We relate our work to Birkhoff’s variety theorem [MT92], which states that a class \mathbf{A} of algebras can be characterized by equational axioms only up to its variety closure $\text{vc}(\mathbf{A})$. If \mathbf{A} is a finite class of finite algebras such that $\text{vc}(\mathbf{A})$ is finitely axiomatizable at all, we can compute an equational axiom set for it (Cor. 15).

As an application in the field of automated theorem proving, we give a sufficient criterion for establishing whether a class \mathbf{A} of algebras is a *prototype* class for a given theory Θ (Cor. 17). If the criterion applies, the validity of any formula φ in n variables can be decided quickly and simply by merely testing φ in \mathbf{A} , avoiding the search space of usual theorem proving procedures: $\Theta \models \varphi$ if and only if φ is satisfied in every $\mathcal{A} \in \mathbf{A}$.

Section 2 recalls some formal definitions. In order to make this paper self-contained, we refer well-known results on regular tree grammars that are used in the sequel. Section 3 first gives a rational reconstruction of Barzdins' liquid flow algorithm; then we show how to compute an axiom set for a finite class of finite algebras. In Sect. 4 and 5, we discuss the applications to Birkhoff characterizations and prototype algebras in theorem proving, respectively. A full version including all proofs can be found in [Bur02].

2 Definitions and Notations

Definition 1. *[Sorted term, substitution]* We assume familiarity with the classic definitions of terms and substitutions in a many-sorted framework. Let \mathcal{S} be a finite set of sorts. A signature Σ is a set of function symbols f , each of which has a fixed domain and range. Let \mathcal{V} be an infinite set of variables, each of a fixed sort. For $S \in \mathcal{S}$ and $V \subseteq \mathcal{V}$, $\mathcal{T}_{V,S}(\Sigma)$ denotes the set of all well-sorted terms of sort S over Σ and V ; let $\mathcal{T}_V(\Sigma) := \bigcup_{S \in \mathcal{S}} \mathcal{T}_{V,S}(\Sigma)$. Let $\text{sort}(t)$ denote the unique sort of a term t . $\sigma = \{x_1 \mapsto t_1, \dots, x_n \mapsto t_n\}$ denotes a well-sorted substitution that maps each variable x_i to the term t_i . \square

Definition 2. *[Algebra]* We consider w.l.o.g. term algebras factorized by a set of operation-defining equations. In this setting, a finite many-sorted algebra \mathcal{A} of signature Σ is given by a nonempty finite set \mathcal{A}_S of constants for each sort $S \in \mathcal{S}$ and a set $E_{\mathcal{A}}$ consisting of exactly one equation $f(a_1, \dots, a_n) = a$ for each $f \in \Sigma$ with $f : S_1 \times \dots \times S_n \rightarrow S$ and each $a_1 \in \mathcal{A}_{S_1}, \dots, a_n \in \mathcal{A}_{S_n}$, where $a \in \mathcal{A}_S$. The \mathcal{A}_S are just the domains of \mathcal{A} for each sort S , while $E_{\mathcal{A}}$ defines the operations from Σ on these domains. Define $\Sigma_{\mathcal{A}} := \Sigma \cup \bigcup_{S \in \mathcal{S}} \mathcal{A}_S$. We write $(=_{\mathcal{A}})$ for the congruence relation induced by $E_{\mathcal{A}}$; each ground term $t \in \mathcal{T}_{\{\},S}(\Sigma_{\mathcal{A}})$ equals exactly one $a \in \mathcal{A}_S$. \square

We will only allow closed quantified equations as formulas. This is sufficient since an arbitrary formula can always be transformed into prenex normal form, and we can model predicates and junctors by functions into a dedicated sort *Bool*.

Definition 3. *[Formula, theory]* For a k -tuple $\mathbf{x} = \langle x_1, \dots, x_k \rangle \in \mathcal{V}^k$ such that $x_i \neq x_j$ for $i \neq j$, define $\mathcal{Q}(\mathbf{x}) := \{q_1 x_1 \dots q_k x_k \mid q_1, \dots, q_k \in \{\forall, \exists\}\}$ as the set of all quantifier prefixes over \mathbf{x} . Any expression of the form $Q : t_1 =_S t_2$ for $Q \in \mathcal{Q}(\mathbf{x})$ and $t_1, t_2 \in \mathcal{T}_{\mathbf{x},S}(\Sigma)$ is called a formula over Σ and \mathbf{x} . We will sometimes omit the index of $(=_S)$. We denote a formula by φ , and a set of formulas, also called theory, by Θ . \square

When encoding predicates and junctors using a sort *Bool*, in order to obtain an appropriate semantics¹ it is necessary and sufficient to fix the interpretation of the sort *Bool* accordingly for every algebra under consideration. Therefor, we define below the notion of an *admitted algebra*, and let the definition of **sat**, \models , etc. depend on it.

We tacitly assume that,

- when we consider only equations, each algebra is admitted, while,
- when we consider arbitrary predicates, junctors and a sort *Bool*, only algebras with an appropriate interpretation of *Bool* are admitted.

Definition 4. [Admitted algebras] Let a signature Σ be given. Let $\mathcal{S}_{\text{fix}} \subseteq \mathcal{S}$ be a set of sorts; and let Σ_{fix} be the set of all $f \in \Sigma$ that have all argument and result sorts in \mathcal{S}_{fix} . Let a fixed Σ_{fix} -algebra \mathcal{A}_{fix} be given; we denote its domain sets by $\mathcal{A}_{\text{fix},S}$.

We say that a Σ -algebra \mathcal{A} is admitted if $\mathcal{A}_S = \mathcal{A}_{\text{fix},S}$ for each $S \in \mathcal{S}_{\text{fix}}$ and $t_1 =_{\mathcal{A}} t_2 \Leftrightarrow t_1 =_{\mathcal{A}_{\text{fix}}} t_2$ for all $t_1, t_2 \in \mathcal{T}_{\{\},S}(\Sigma_{\mathcal{A}})$ and $S \in \mathcal{S}_{\text{fix}}$. \square

Definition 5. [Validity] For an admitted Σ -algebra \mathcal{A} and a formula φ , we write $\mathcal{A} \text{ sat } \varphi$ if φ is valid in \mathcal{A} , where equality symbols $(=_S)$ in φ are interpreted as identity relations on \mathcal{A}_S , rather than by an arbitrary congruence on it. For a class of admitted Σ -algebras \mathbf{A} , and a theory Θ , we similarly write $\mathbf{A} \text{ sat } \varphi$, $\mathcal{A} \text{ sat } \Theta$, and $\mathbf{A} \text{ sat } \Theta$. Define $\Theta_1 \models \Theta_2$ if $\mathcal{A} \text{ sat } \Theta_1$ implies $\mathcal{A} \text{ sat } \Theta_2$ for each admitted Σ -algebra \mathcal{A} .

If we choose $\mathcal{S}_{\text{fix}} := \{\}$, each algebra is admitted. Choosing $\mathcal{S}_{\text{fix}} := \{\text{Bool}\}$, $\Sigma_{\text{fix}} := \{(\neg), (\wedge), (\vee), (\rightarrow), (\leftrightarrow)\}$, and \mathcal{A}_{fix} as the two-element Boolean algebra, we prescribe the interpretation of *Bool* for each admitted algebra. \square

Definition 6. [Complete theorem sets] For a Σ -algebra \mathcal{A} , and a tuple \mathbf{x} of variables as in Def. 3, define $\mathcal{TH}_{\mathbf{x}}(\mathcal{A}) :=$

$$\{Q : t_1 =_S t_2 \mid Q \in \mathcal{Q}(\mathbf{x}), S \in \mathcal{S}, t_1, t_2 \in \mathcal{T}_{\mathbf{x},S}(\Sigma), \mathcal{A} \text{ sat } (Q : t_1 =_S t_2)\}$$

as the set of all formulas over Σ and \mathbf{x} that are valid in \mathcal{A} . The elements of $\mathcal{TH}_{\mathbf{x}}(\mathcal{A})$ can be considered as terms over the extended signature

$$\Sigma \cup \{ (=_S) \mid S \in \mathcal{S} \} \cup \{ (Q :) \mid Q \in \mathcal{Q}(\mathbf{x}) \}.$$

For a class \mathbf{A} of Σ -algebras, define $\mathcal{TH}_{\mathbf{x}}(\mathbf{A}) := \bigcap_{\mathcal{A} \in \mathbf{A}} \mathcal{TH}_{\mathbf{x}}(\mathcal{A})$. \square

Example 7. The algebra \mathcal{A}_2 , defined by $\mathcal{A}_{\text{Nat}} := \{0, 1\}$ and $E_{\mathcal{A}} = \{0+0=0, 0+1=1, 1+0=1, 1+1=0\}$, is a Σ -algebra for $\Sigma = \{0, (+)\}$. The set $\mathcal{TH}_{\langle x,y \rangle}(\mathcal{A}_2)$ contains the formula $\forall x \exists y : x+y=0$, but not $\forall x \exists y : x+y=1$, since $1 \notin \Sigma$. \square

¹ If in some algebra \mathcal{A} we had $\mathcal{A}_{\text{Bool}} = \{a\}$ and $(\text{true} = a), (\text{false} = a) \in E_{\mathcal{A}}$, any formula φ was valid in \mathcal{A} .

Definition 8. [Regular tree grammar] A regular tree grammar \mathcal{G} consists of rules $N ::= f_1(N_{11}, \dots, N_{1n_1}) \mid \dots \mid f_m(N_{m1}, \dots, N_{mn_m})$ or $N ::= N_1 \mid \dots \mid N_m$ where N, N_i, N_{ij} are nonterminal symbols and $f_i \in \Sigma$. Note that n_i may be also 0. Each $f_i(N_{i1}, \dots, N_{in_i})$ or N_i is called an alternative. N, N_i, N_{ij} are assigned a sort each that have to fit with each other and with f_i .

The size $|\mathcal{G}|$ of \mathcal{G} is its total number of alternatives. We denote the set of nonterminals of \mathcal{G} by \mathcal{N} . The language produced by a nonterminal N of \mathcal{G} is denoted by $\mathcal{L}_{\mathcal{G}}(N)$, it is a set of ground terms over Σ ; if N is the start symbol of \mathcal{G} , we also write $\mathcal{L}(\mathcal{G})$. \mathcal{G} is called deterministic if no different rules have identical alternatives.

Define the generalized height $hg(t)$ of a ground term t by

$$hg(f(t_1, \dots, t_n)) := \max\{hg(t_1), \dots, hg(t_n)\} + hg(f),$$

where $\max\{\} := 0$, and $hg(f) \in \mathbb{N}$ may be defined arbitrarily. For a nonterminal N of a grammar \mathcal{G} , define $hg(N)$ as the minimal height of any term in $\mathcal{L}_{\mathcal{G}}(N)$, it is ∞ if $\mathcal{L}_{\mathcal{G}}(N)$ is empty. \square

Theorem 9. [Properties of regular tree grammars]

1. Incorporating [McA92, Sect.6]

Given a finite many-sorted Σ -algebra \mathcal{A} , a grammar $\mathcal{G} = \text{incorporate}(\mathcal{A})$ of size $|E_{\mathcal{A}}|$ can be computed in time $\mathcal{O}(|E_{\mathcal{A}}|)$ such that

$$\forall S \in \mathcal{S}, a \in \mathcal{A}_S \exists N_a \in \mathcal{N} : \mathcal{L}_{\mathcal{G}}(N_a) = \{t \in \mathcal{T}_{\Sigma}(\Sigma_{\mathcal{A}}) \mid t =_{\mathcal{A}} a\}.$$

2. Externing [McA92, Sect.3]

Given a deterministic grammar \mathcal{G} , a set E of $|\mathcal{G}|$ ground equations can be computed in time $\mathcal{O}(|\mathcal{G}|)$ such that for all ground terms t_1, t_2 :

$$t_1 =_E t_2 \Leftrightarrow \exists N \in \mathcal{N} : t_1, t_2 \in \mathcal{L}_{\mathcal{G}}(N),$$

where $(=_E)$ denotes the congruence induced by E .

3. Lifting [CDG⁺99, Thm.7 in Sect.1.4]

Given a grammar \mathcal{G} and a ground substitution σ , a grammar $\mathcal{G}' = \text{lift}(\mathcal{G}, \sigma)$ of size $|\mathcal{G}| + |\mathcal{N}| \cdot |\text{dom } \sigma|$ can be computed in time $\mathcal{O}(|\mathcal{G}'|)$ such that

$$\forall N \in \mathcal{N} \exists N' \in \mathcal{N}' : \mathcal{L}_{\mathcal{G}'}(N') = \{t \in \mathcal{T}_{\text{dom } \sigma} \mid \sigma t \in \mathcal{L}_{\mathcal{G}}(N)\},$$

where \mathcal{N} and \mathcal{N}' is the set of nonterminals of \mathcal{G} and \mathcal{G}' , respectively. Note that the signature gets extended by $\text{dom } \sigma$; these variables are treated as constants in \mathcal{G}' .

4. Intersection [CDG⁺99, Sect.1.3]

Given n grammars $\mathcal{G}_1, \dots, \mathcal{G}_n$, a grammar $\mathcal{G} = \text{intersect}(\mathcal{G}_1, \dots, \mathcal{G}_n)$ of size $|\mathcal{G}_1| \cdot \dots \cdot |\mathcal{G}_n|$ can be computed in time $\mathcal{O}(|\mathcal{G}|)$ such that for each

$$\forall N_{i_1} \in \mathcal{N}_1, \dots, N_{i_n} \in \mathcal{N}_n \exists N_{i_1, \dots, i_n} \in \mathcal{N} : \mathcal{L}_{\mathcal{G}}(N) = \bigcap_{j=1}^n \mathcal{L}_{\mathcal{G}_j}(N_{i_j}).$$

5. Restriction [special case of 4]

Intersection of one grammar \mathcal{G}_1 with a term universe $\mathcal{T}_{V,S}(\Sigma)$, such that

$$\forall N_1 \in \mathcal{N}_1 \exists N \in \mathcal{N} : \mathcal{L}_{\mathcal{G}}(N) = \mathcal{L}_{\mathcal{G}_1}(N_1) \cap \mathcal{T}_{V,S}(\Sigma)$$

can be done in time $\mathcal{O}(|\mathcal{G}_1|)$ by removing all symbols not in Σ .

6. Union [CDG⁺99, Sect.1.3]

Given n grammars $\mathcal{G}_1, \dots, \mathcal{G}_n$, a grammar $\mathcal{G} = \text{unite}(\mathcal{G}_1, \dots, \mathcal{G}_n)$ of size $n + |\mathcal{G}_1| + \dots + |\mathcal{G}_n|$ can be computed in time $\mathcal{O}(n)$ such that

$$\mathcal{L}(\mathcal{G}) = \bigcup_{i=1}^n \mathcal{L}(\mathcal{G}_i)$$

by adding one rule.

7. Composition [Trivial]

Given an n -ary function symbol f , a grammar \mathcal{G} , and nonterminals N_1, \dots, N_n , a grammar $\mathcal{G}' = \text{tag}(\mathcal{G}, f(N_1, \dots, N_n))$ of size $|\mathcal{G}| + 1$ can be computed in time $\mathcal{O}(1)$ such that

$$\mathcal{L}_{\mathcal{G}'}(N) = \{f(t_1, \dots, t_n) \mid \bigwedge_{i=1}^n t_i \in \mathcal{L}_{\mathcal{G}}(N_i)\}$$

for a certain nonterminal N , by adding one rule.

8. Weight computation [AM91, Sect.4]

Given a grammar \mathcal{G} , the heights $hg(N)$ can be computed for all nonterminals $N \in \mathcal{N}$ simultaneously in time $\mathcal{O}(|\mathcal{N}|^2)$.

9. Language enumeration [BH96, Fig.21]

Given a grammar \mathcal{G} and the heights of all nonterminals, the elements of $\mathcal{L}_{\mathcal{G}}(N)$ can be enumerated in order of increasing height in time linear in the sum of their sizes by a simple PROLOG program. \square

3 Equational Theories of Finite Algebras

First, we give a rational reconstruction of the *liquid flow algorithm* of Barzdin and Barzdin [BB91]. They use labeled graphs to compute an axiom set from an incompletely given finite algebra. Their approach can be reformulated in terms of regular tree languages using the correspondence of notions shown in Fig. 1. Our following theorem corresponds to their main result, Thm. 2. It is in fact a slight extension, as it allows for sorts and for substitutions that map several variables to the same value.

On the other hand, the *liquid flow algorithm* turns out to be an improvement of the weight computation algorithm from Thm. 9.8. Both are fixpoint algorithms, and identical except for minor, but important, modifications. The algorithm from Thm. 9.8 has a complexity of $\mathcal{O}(|\mathcal{N}|^2)$, while Barzdins' algorithm runs in $\mathcal{O}(|\mathcal{N}|)$, exploiting the fact that always $hg(f) \leq 1$ and therefore the first value $< \infty$ assigned to some $hg(N)$ must be its final one already. Since

in each cycle at least one N must change its assigned hg value², by an appropriate incremental technique (*water front*), linear complexity can be achieved. A formal complexity proof of this improved grammar fixpoint algorithm, extended to somewhat more general weight definitions, shall appear in [Bur02].

Theorem 10. [*Reconstruction of Barzdin*] Given a Σ -algebra \mathcal{A} , domain elements $b_1, \dots, b_n \in \mathcal{A}_{S_0}, a_{11}, \dots, a_{n1} \in \mathcal{A}_{S_1}, \dots, a_{1k}, \dots, a_{nk} \in \mathcal{A}_{S_k}$, and defining $\sigma_i = \{x_1 \mapsto a_{i1}, \dots, x_k \mapsto a_{ik}\}$ for $i = 1, \dots, n$ and $V := \{x_1, \dots, x_k\}$, the set of terms $T = \{t \in \mathcal{T}_{V, S_0}(\Sigma) \mid \bigwedge_{i=1}^n \sigma_i t =_{\mathcal{A}} b_i\}$ is a regular tree language. A grammar for it can be computed in time $\mathcal{O}(|E_{\mathcal{A}}|^n)$. After computing nonterminal weights in time $\mathcal{O}(|E_{\mathcal{A}}|^{2n})$, the language elements can be enumerated in order of increasing height in linear time.

Proof. Using the notions of Thm. 9, let $\mathcal{G}_0 := \text{incorporate}(\mathcal{A})$, and $\mathcal{G}_i := \text{lift}(\mathcal{G}_0, \sigma_i)$ for $i = 1, \dots, n$. We have

$$\mathcal{L}_{\mathcal{G}_i}(N_a) = \{t \in \mathcal{T}_V(\Sigma_A) \mid \sigma_i t =_{\mathcal{A}} a\}.$$

Let $\mathcal{G} := \text{intersect}(\mathcal{G}_1, \dots, \mathcal{G}_n, \mathcal{T}_V(\Sigma))$, then

$$\mathcal{L}_{\mathcal{G}}(N_{a_1, \dots, a_n}) = \{t \in \mathcal{T}_V(\Sigma) \mid \bigwedge_{i=1}^n \sigma_i t =_{\mathcal{A}} a_i\}.$$

Hence $T = \mathcal{L}_{\mathcal{G}}(N_{b_1, \dots, b_n})$. Note that \mathcal{G} itself does not depend on b_1, \dots, b_n . Compute the height of all nonterminals of \mathcal{G} using Thm. 9.8, using $hg(x_i) := 0$ for $x_i \in V$ and $hg(f) := 1$ for $f \in \Sigma$. Use Thm. 9.9 to enumerate the terms of $\mathcal{L}_{\mathcal{G}}(N_{b_1, \dots, b_n})$. \square

Barzdin and Barzdin allow to specify an algebra incompletely, since their main goal is to acquire *hypotheses* about what are probable axioms.

We now investigate the special case that the substitutions $\sigma_1, \dots, \sigma_n$ in Thm. 10 describe *all* possible assignments of algebra domain elements to the variables x_1, \dots, x_k . This way, we obtain *certainty* about the computed axioms – they are guaranteed to be valid in the given algebra.

Theorem 11. [*Computing complete theorem sets*] Let $\mathbf{x} = \langle x_1, \dots, x_k \rangle$ be a k -tuple of variables, \mathcal{A} be a finite Σ -algebra and \mathbf{A} a finite class of finite Σ -algebras, then $\mathcal{TH}_{\mathbf{x}}(\mathcal{A})$ and $\mathcal{TH}_{\mathbf{x}}(\mathbf{A})$ are regular tree languages.

Proof (sketch). Define $\mathcal{A}_{\mathbf{x}} := \mathcal{A}_{\text{sort}(x_1)} \times \dots \times \mathcal{A}_{\text{sort}(x_k)}$. For each $\mathbf{a} \in \mathcal{A}_{\mathbf{x}}$, let $\sigma_{\mathbf{a}} := \{x_1 \mapsto a_1, \dots, x_k \mapsto a_k\}$. Let $\mathcal{G} = \text{incorporate}(\mathcal{A})$ and $\mathcal{G}_{\mathbf{a}} = \text{intersect}(\text{lift}(\mathcal{G}, \sigma_{\mathbf{a}}), \mathcal{T}_{\mathbf{x}}(\Sigma))$ for $\mathbf{a} \in \mathcal{A}_{\mathbf{x}}$. For $S \in \mathcal{S}$ and $a \in \mathcal{A}_S$, let $\mathcal{G}_{\mathbf{a}, a} = \text{tag}(\mathcal{G}_{\mathbf{a}}, N_a =_S N_a)$, where $(=_S)$ is a new binary infix function symbol. Let $\mathcal{G}'_{\mathbf{a}} = \text{unite}(\{\mathcal{G}_{\mathbf{a}, a} \mid S \in \mathcal{S}, a \in \mathcal{A}_S\})$ for each $\mathbf{a} \in \mathcal{A}_{\mathbf{x}}$. We have

$$\mathcal{L}(\mathcal{G}'_{\mathbf{a}}) = \{t_1 =_S t_2 \mid S \in \mathcal{S}, t_1, t_2 \in \mathcal{T}_{\mathbf{x}, S}(\Sigma), \sigma_{\mathbf{a}} t_1 =_{\mathcal{A}} \sigma_{\mathbf{a}} t_2\}.$$

² Unless the fixpoint has been reached already

Barzdin [BB91]	Tree Grammars
sample P	equations $E_{\mathcal{A}}$ from Def. 2
open term, level	term in $\mathcal{T}_{\mathcal{V}}(\Sigma)$, height
closed term	term in $\mathcal{T}_{\{\}}(\Sigma_{\mathcal{A}})$
sample graph	grammar $\mathcal{G} = \text{incorporate}(\mathcal{A})$
domain node	nonterminal N_a
functional node	expression $f(N_{a_1}, \dots, N_{a_n})$
upper node	N_a , if $N_a ::= \dots f(N_{a_1}, \dots, N_{a_n}) \dots$
lower nodes	N_{a_1}, \dots, N_{a_n}
node weight	language height $hg(N)$ from Def. 8
chain of dotted arcs	rule rhs with alternatives ordered by increasing height
annotated sample graph	grammar with heights of nonterminals obtained from Thm. 9.8
liquid-flow algorithm	(improved) height computation algorithm from Thm. 9.8
α -term	term in $\mathcal{T}_{\{\}}(\Sigma_{\mathcal{A}}) \cap \bigcup_{a \in \mathcal{A}_S} \mathcal{L}(N_a)$
minimal α -term of domain node d	term $t \in \mathcal{L}(N_d)$ of minimal height
minimal α -term of functional node	term $t \in \mathcal{L}(f(N_{a_1}, \dots, N_{a_n}))$ of minimal height
Theorem 2	Theorem 10
Theorem 1	Theorem 10 for $n = 1$

Fig. 1. Correspondence of notions between [BB91] and regular tree grammars

Now, for each $Q \in \mathcal{Q}(\mathbf{x})$, apply set operations corresponding to Q to the \mathcal{G}_a ; e.g. if $k = 2$ and $Q = (\forall x_1 \exists x_2)$, let

$$\mathcal{G}_Q = \text{intersect}(\{\text{unite}(\{\mathcal{G}_{a_1, a_2} \mid a_2 \in \mathcal{A}_{\text{sort}(a_2)}\}) \mid a_1 \in \mathcal{A}_{\text{sort}(a_1)}\}).$$

In general, we get

$$\mathcal{L}(\mathcal{G}_Q) = \{t_1 =_S t_2 \mid S \in \mathcal{S}, t_1, t_2 \in \mathcal{T}_{\mathbf{x}, S}(\Sigma), \mathcal{A} \text{ sat } (Q : t_1 = t_2)\}.$$

For each $Q \in \mathcal{Q}(\mathbf{x})$, let $\mathcal{G}'_Q = \text{tag}(\mathcal{G}_Q, Q : \mathcal{G}_Q)$, where $(Q :)$ is a new unary prefix function symbol. Let $\mathcal{G}' = \text{unite}(\{\mathcal{G}'_Q \mid Q \in \mathcal{Q}(\mathbf{x})\})$, then $\mathcal{TH}_{\mathbf{x}}(\mathcal{A}) = \mathcal{L}(\mathcal{G}')$. From this, we immediately get a grammar for $\mathcal{TH}_{\mathbf{x}}(\mathbf{A})$ by Thm. 9.4. \square

Example 12. Let us compute $\mathcal{TH}_{\mathbf{x}}(\mathcal{A}_2)$ for the algebra \mathcal{A}_2 from Exm. 7 and $\mathbf{x} = \langle x, y \rangle$. We have the substitutions $\sigma_{00}, \sigma_{01}, \sigma_{10}, \sigma_{11}$ and use the naming convention

$$\mathcal{L}(N_{ijkl}) = \mathcal{L}(\mathcal{G}_{00, i}) \cap \mathcal{L}(\mathcal{G}_{01, j}) \cap \mathcal{L}(\mathcal{G}_{10, k}) \cap \mathcal{L}(\mathcal{G}_{11, l}),$$

e.g. $\mathcal{L}(N_{0011}) = \{t \in \mathcal{T}_{\mathbf{x}}(\Sigma) \mid \mathcal{A}_2 \text{ sat } (\sigma_{00}t = \sigma_{01}t = 0 \wedge \sigma_{10}t = \sigma_{11}t = 1)\}$. A “*” may serve as *don't care symbol*, e.g. $\mathcal{L}(N_{i**k*}) = \mathcal{L}(\mathcal{G}_{00, i}) \cap \mathcal{L}(\mathcal{G}_{10, k})$.

From Thm. 11, after incorporating, lifting, and restricting, we obtain e.g. \mathcal{G}_{00} , with nonterminals N_{0***} and N_{1***} . As $\mathcal{L}(N_{1***})$ turns out to be empty, we simply have

$$N_{0***} ::= 0 \mid x \mid y \mid N_{0***} + N_{0***}.$$

$N_{0000} ::= 0$	$ $	$N_{0000} + N_{0000}$	$ $	$N_{0011} + N_{0011}$	$ $	$N_{0101} + N_{0101}$	$ $	$N_{0110} + N_{0110}$	
$N_{0011} ::= x$	$ $	$N_{0000} + N_{0011}$	$ $	$N_{0011} + N_{0000}$	$ $	$N_{0101} + N_{0110}$	$ $	$N_{0110} + N_{0101}$	
$N_{0101} ::= y$	$ $	$N_{0000} + N_{0101}$	$ $	$N_{0011} + N_{0110}$	$ $	$N_{0101} + N_{0000}$	$ $	$N_{0110} + N_{0011}$	
$N_{0110} ::=$		$N_{0000} + N_{0110}$	$ $	$N_{0011} + N_{0101}$	$ $	$N_{0101} + N_{0011}$	$ $	$N_{0110} + N_{0000}$	
$N_{0*0*} ::=$		N_{0000}	$ $	N_{0001}	$ $	N_{0100}	$ $	N_{0101}	
$N_{0*1*} ::= \dots$									
$N_{\forall\forall} ::=$		$N_{0000} = N_{0000}$	$ $	$N_{0011} = N_{0011}$	$ $	$N_{0101} = N_{0101}$	$ $	$N_{0110} = N_{0110}$	
$N_{\forall\exists} ::=$		$N_{0*0*} = N_{0*0*}$	$ $	$N_{0*1*} = N_{0*1*}$	$ $	$N_{0**0} = N_{0**0}$	$ $	$N_{0**1} = N_{0**1}$	
		$ $	$N_{*00*} = N_{*00*}$	$ $	$N_{*01*} = N_{*01*}$	$ $	$N_{*10*} = N_{*10*}$	$ $	$N_{*11*} = N_{*11*}$
		$ $	$N_{*0*0} = N_{*0*0}$	$ $	$N_{*0*1} = N_{*0*1}$	$ $	$N_{*1*0} = N_{*1*0}$	$ $	$N_{*1*1} = N_{*1*1}$
$N_{\exists\forall} ::=$		$N_{00**} = N_{00**}$	$ $	$N_{01**} = N_{01**}$					
		$ $	$N_{**00} = N_{**00}$	$ $	$N_{**01} = N_{**01}$	$ $	$N_{**10} = N_{**10}$	$ $	$N_{**11} = N_{**11}$
$N_{\exists\exists} ::=$		$N_{0***} = N_{0***}$	$ $	$N_{*0**} = N_{*0**}$	$ $	$N_{*1**} = N_{*1**}$			
		$ $	$N_{**0*} = N_{**0*}$	$ $	$N_{**1*} = N_{**1*}$	$ $	$N_{***0} = N_{***0}$	$ $	$N_{***1} = N_{***1}$
$N ::=$		$\forall x\forall y : N_{\forall\forall}$	$ $	$\forall x\exists y : N_{\forall\exists}$	$ $	$\exists x\forall y : N_{\exists\forall}$	$ $	$\exists x\exists y : N_{\exists\exists}$	

Fig. 2. Grammar \mathcal{G}' for $\mathcal{TH}_x(\mathcal{A})$ in Exm. 12

N		N	
$\forall x\forall y :$	$\overline{N_{\forall\forall}}$	$\forall x\exists y :$	$\overline{N_{\forall\exists}}$
$\forall x\forall y :$	$\overline{N_{0110}} = \overline{N_{0110}}$	$\forall x\exists y :$	$\overline{N_{0**0}} = \overline{N_{0**0}}$
$\forall x\forall y :$	$\overline{N_{0011} + N_{0101}} = \overline{N_{0101} + N_{0011}}$	$\forall x\exists y :$	$\overline{N_{0110}} = \overline{N_{0000}}$
$\forall x\forall y :$	$\overline{x + y} = \overline{y + x}$	$\forall x\exists y :$	$\overline{N_{0011} + N_{0101}} = \overline{0}$
		$\forall x\exists y :$	$\overline{x + y} = \overline{0}$

Fig. 3. Example derivations from \mathcal{G}' in Exm. 12

We obtain \mathcal{G}'_{00} by just adding the rule $N_{=} ::= (N_{0***} = N_{0***})$. To compute $\mathcal{G}_{\forall\forall}$, we build all intersections N_{ijkl} without “*”; only four of them turn out to be nonempty, their rules are shown in Fig. 2. The grammar $\mathcal{G}_{\forall\forall}$ consists of these rules and an additional one for its start symbol $N_{\forall\forall}$. The grammars $\mathcal{G}_{\forall\exists}$, $\mathcal{G}_{\exists\forall}$, and $\mathcal{G}_{\exists\exists}$ have similar starting rules, which use only nonterminals N_{ijkl} containing a “*”. Since the rules for the latter are trivial, only the one for N_{0*0*} is shown. Finally, the grammar \mathcal{G}' for $\mathcal{TH}_x(\mathcal{A}_2)$ consists of all these rules and an additional one for its start symbol N . Figure 3 show some example derivations. \square

Theorem 13. *[Computing complete axiom sets] The sets $\mathcal{TH}_x(\mathcal{A})$ and $\mathcal{TH}_x(\mathbf{A})$ obtained from Thm. 11 can be represented as the deductive closure of a finite set of formulas, called $\mathcal{AX}_x(\mathcal{A})$ and $\mathcal{AX}_x(\mathbf{A})$, respectively. We have:*

$$\begin{aligned} \forall t_1, t_2 \in \mathcal{T}_x(\Sigma) : \mathcal{A} \text{ sat } t_1 = t_2 &\Leftrightarrow \mathcal{AX}_x(\mathcal{A}) \models t_1 = t_2, \text{ and} \\ \forall t_1, t_2 \in \mathcal{T}_x(\Sigma) : \mathbf{A} \text{ sat } t_1 = t_2 &\Leftrightarrow \mathcal{AX}_x(\mathbf{A}) \models t_1 = t_2. \end{aligned}$$

Proof (sketch). First, we consider the purely universal formulas in $\mathcal{TH}_x(\mathcal{A})$. Using the notions of Thm. 11, the grammar $\mathcal{G}_{\forall\dots\forall}$ is deterministic since no union operations were involved in its construction. Using Thm. 9.2, we get a finite set $E_{\forall\dots\forall}$ of equations, each of which we compose with the appropriate universal quantifier prefix $(\forall x_1 \dots \forall x_k :)$. The resulting formula set $E'_{\forall\dots\forall}$ implies any purely universal equation valid in \mathcal{A} . By construction of $E_{\forall\dots\forall}$, it can reduce each term t in any quantified equation in $\mathcal{TH}_x(\mathcal{A})$ to a unique normal form. Let NF denote the set of all those normal forms; it is finite since $|\mathcal{N}|$ is finite.

Next, for any quantifier prefix Q containing some “ \exists ”, let

$$E'_Q := \{Q : t_{1n} = t_{2n} \mid t_{1n}, t_{2n} \in NF, (t_{1n} = t_{2n}) \in \mathcal{L}(\mathcal{G}_Q), t_{1n} \neq t_{2n}\}.$$

Any formula $Q : t_1 = t_2$ in $\mathcal{L}(\mathcal{G}'_Q)$ can then be deduced from $\forall \dots \forall : t_1 = t_{1n}$ and $\forall \dots \forall : t_2 = t_{2n}$ in $E'_{\forall\dots\forall}$ and $Q : t_{1n} = t_{2n}$ in E'_Q , where t_{1n} and t_{2n} are the normal forms of t_1 and t_2 , respectively.

Finally, let $\mathcal{AX}_x(\mathcal{A}) = \bigcup_{Q \in \mathcal{Q}(x)} E'_Q$. The proof for $\mathcal{AX}_x(\mathbf{A})$ is similar. \square

Observe that the variables in x are introduced as constants into the grammars, hence $E_{\forall\dots\forall}$ in the above proof is a set of ground equations. A closer look at the algorithm referred by Thm. 9.2 reveals that it generates in fact a Noetherian ground-rewriting system assigning unique normal forms. Anyway, no proper instance of any formula from $\mathcal{AX}_x(\mathcal{A})$ is needed to derive any one in $\mathcal{TH}_x(\mathcal{A})$. By permitting proper instantiations, we may delete formulas that are instances of others, thus reducing their total number significantly. To find such subsumed formulas, an appropriate indexing technique may be used, see e.g. [Gra94].

Example 14. Continuing Exm. 12, and referring to the notions the proof of of Thm. 13, we obtain the set $E_{\forall\forall}$ shown at the top of Fig. 4, where we chose the normal form of N_{0000} , N_{0011} , N_{0101} , and N_{0110} as 0, x , y , and $x+y$, respectively³,

³ The algorithm of Thm. 9.2 can easily be modified to work with arbitrary chosen normal forms instead of external constants.

$$\begin{array}{ll}
\forall\forall : & N_{0000} : \boxed{0} = 0 + 0 = \boxed{x+x} = y + y = (x+y) + (x+y) \\
& N_{0011} : \boxed{x} = 0 + x = \boxed{x+0} = y + (x+y) = \boxed{(x+y) + y} \\
& N_{0101} : \boxed{y} = 0 + y = x + (x+y) = \boxed{y+0} = \boxed{(x+y) + x} \\
& N_{0110} : \boxed{x+y} = 0 + (x+y) = \boxed{y+x} = (x+y) + 0 \\
\\
\forall\exists : & N_{0*0*} : 0 = y \quad N_{0**0} : \boxed{0 = x+y} \quad N_{0*1*} : x = x+y \quad N_{0**1} : x = y \\
\exists\forall : & N_{00**} : 0 = x \quad N_{01**} : y = x+y \\
\exists\exists : & N_{0***} : 0 = x = y = x+y \quad N_{*0**} : 0 = x \quad N_{*1**} : y = x+y \\
& N_{**0*} : 0 = y \quad N_{**1*} : x = x+y \quad N_{***0} : 0 = x+y \quad N_{***1} : x = y
\end{array}$$

Fig. 4. Axioms $\mathcal{AX}_x(\mathcal{A})$ in Exm. 14

which are each of minimal size. From each rule alternative in Fig. 2, we get one equation that is universally valid in \mathcal{A} . The equations between marked terms remain nontrivial if instantiations are allowed.

For each of the N_{ijkl} with a “*”, we check which of the above 4 normal forms are member of $\mathcal{L}(N_{ijkl})$. This can be decided quickly by matching the index; e.g. N_{0*0*} contains 0 and y since $0*0*$ matches with 0000 and 0101. Each pair of normal forms in the same $\mathcal{L}(N_{ijkl})$ gives rise to an equation, shown at the bottom of Fig. 4. Note that equations between terms of different N_{ijkl} are neither in $\mathcal{TH}_x(\mathcal{A})$ nor in $\mathcal{AX}_x(\mathcal{A})$. For example, “crossing” N_{0*0*} and N_{0**0} yields the forbidden formula $\forall x \exists y : y = x + y$ which does not hold in \mathcal{A} .

After removing all redundancies⁴, we get

$$\left\{ \begin{array}{l} \forall x : 0 = x+x, \quad \forall x \forall y : x+y = y+x, \quad \forall x \exists y : 0 = x+y, \\ \forall x : x = x+0, \quad \forall x \forall y : x = (x+y)+y \end{array} \right\}$$

as a set of formulas implying every closed formula over $\{x, y, 0, (+), (=)\}$ that is valid in \mathcal{A} . Note that the associativity law is not implied, since it requires 3 variables. \square

4 Application to Equational Theories

We now show some consequences of axiomatization properties in a purely equational setting. Remember our convention made before Def. 4, that in this setting, each algebra is considered to be admitted. We restrict \mathcal{TH} and \mathcal{AX} to the set

$$\mathcal{U} := \{\forall \dots \forall : t_1 =_S t_2 \mid S \in \mathcal{S}, t_1, t_2 \in \mathcal{T}_{\mathcal{V}, S}(\Sigma)\},$$

which is trivially a regular tree language.

For a given signature Σ and a class \mathbf{A} of Σ -algebras, let $\text{vc}(\mathbf{A})$ denote the smallest variety containing \mathbf{A} , i.e., the class of all Σ -algebras obtainable from algebras in \mathbf{A} by building subalgebras, Cartesian products, and homomorphic

⁴ E.g. $\exists x \forall y : y = x + y$ follows from $\forall x : x = x + 0$ and $\forall x \forall y : x + y = y + x$.

images. For a set E of equations, let $\text{Mod}(E)$ denote the class of all Σ -algebras \mathcal{A} with $\mathcal{A} \text{ sat } E$. From Birkhoff's variety theorem [MT92], it is well known that each class \mathbf{A} of algebras can be characterized by universal equations only up to its variety closure $\text{vc}(\mathbf{A})$. However, it is not clear in general how to find such an axiom set E with $\text{Mod}(E) = \text{vc}(\mathbf{A})$.

If \mathbf{A} is a finite class of finite algebras, we can at least construct an increasing sequence of tree languages characterizing $\text{vc}(\mathbf{A})$ *in the limit* (Cor. 15). Whenever there exists any finite axiom set E for $\text{vc}(\mathbf{A})$ at all, the sequence of corresponding model classes eventually becomes stationary, and, using Thm. 13, we can obtain a finite axiom set that uniquely characterizes $\text{vc}(\mathbf{A})$. However, convenient criteria for detecting if and when the sequence becomes stationary are still unknown.

Corollary 15. *[Variety Characterization] For any finite class of finite algebras \mathbf{A} , we can compute a sequence $TH_1 \subseteq TH_2 \subseteq \dots$ of sets of universal equations such that $\text{vc}(\mathbf{A}) = \text{Mod}(\bigcup_{i=1}^{\infty} TH_i)$. If $\text{vc}(\mathbf{A}) = \text{Mod}(E)$ for any finite E , we already have $\text{Mod}(TH_n) = \text{vc}(\mathbf{A})$ for some $n \in \mathbb{N}$. In this case, we can compute a finite axiom set for $\text{vc}(\mathbf{A})$ from TH_n . \square*

Proof. Assuming $\mathcal{V} = \{x_1, x_2, \dots\}$, let $\mathbf{x}_i := \langle x_1, \dots, x_n \rangle$ for $i \in \mathbb{N}$. Let $TH_i := \mathcal{TH}_{\mathbf{x}_i}(\mathbf{A}) \cap \mathcal{U}$ and $TH_{\infty} := \bigcup_{i=1}^{\infty} TH_i$; then, $TH_i \subseteq TH_{i+1} \subseteq TH_{\infty}$. By Thm. 11, TH_{∞} consists of all universal equations that hold in \mathbf{A} .

By Birkhoff's variety theorem, $\text{vc}(\mathbf{A}) = \text{Mod}(E)$ for some set E of equations. Since $\mathbf{A} \text{ sat } E$, we have $E \subseteq TH_{\infty}$, hence $\text{Mod}(TH_{\infty}) \subseteq \text{Mod}(E) = \text{vc}(\mathbf{A})$. Vice versa, we have $\text{vc}(\mathbf{A}) \subseteq \text{Mod}(TH_{\infty})$, since $\mathbf{A} \subseteq \text{Mod}(TH_{\infty})$, and $\text{Mod}(TH_{\infty})$ is closed wrt. subalgebras, products, and homomorphic images.

If E is finite, let $n \in \mathbb{N}$ such that all variables in E occur in \mathbf{x}_n , then $\text{Mod}(TH_n) = \text{vc}(\mathbf{A})$ as above. \square

5 Application to Theorem Proving

Next, we extend our results to arbitrary formulas of first-order predicate logic. This can easily be achieved by including a sort *Bool* and encoding predicates and junctors as functions to *Bool*. We admit only algebras with an appropriate interpretation of *Bool*, cf. the convention before Def. 4.

Thus, the equation set $\mathcal{TH}_{\mathbf{x}}(\mathcal{A})$, and $\mathcal{AX}_{\mathbf{x}}(\mathcal{A})$ corresponds to the set of all formulas in \mathbf{x} valid in \mathcal{A} , and a finite axiomatization of it, respectively. Moreover, we can arbitrarily restrict the set of junctors that may occur in a formula. Note, however, that we cannot get rid of any equality predicate⁵, as they are core components of our approach, cf. Def. 6. Hence, we cannot compute the set of all Horn formulas valid in a given algebra.

Example 16. As an example of computed predicate-logic axiomatizations, consider $(\mathbb{N} \bmod 2)$ with one function (+) and one predicate (<), where the sort *Bool* is interpreted by the two-element Boolean algebra, as required.

⁵ Logical equivalence in *Bool*

$x+x=0$	$0 < x \wedge 0 < y \wedge 0 < x+y \leftrightarrow \text{false}$	$(0 < x \wedge 0 < y) \vee y < x \leftrightarrow 0 < x$
$0+x=x$	$0 < x \wedge 0 < y \wedge x < y \leftrightarrow \text{false}$	$0 < y \wedge 0 < x+y \leftrightarrow x < y$
$x+0=x$	$0 < x \wedge 0 < y \wedge y < x \leftrightarrow \text{false}$	$x < x+y \leftrightarrow x < y$
$(y+x)+y=x$	$0 < x \wedge x < y \leftrightarrow \text{false}$	$0 < y \wedge (0 < x \vee x < y) \leftrightarrow 0 < y$
$(x+y)+y=x$	$x < y \wedge y < x \leftrightarrow \text{false}$	$(0 < x \wedge 0 < y) \vee x < y \leftrightarrow 0 < y$
$(x+x)+y=y$	$x < 0 \leftrightarrow \text{false}$	$0 < x+y \wedge (0 < x \vee x < y) \leftrightarrow 0 < x+y$
$(x+y)+x=y$	$x < x \leftrightarrow \text{false}$	$x < y \vee y < x \leftrightarrow 0 < x+y$
$y+x=x+y$	$x+y < x \leftrightarrow 0 < x \wedge 0 < y$	$(0 < x \wedge 0 < y) \vee 0 < x+y \leftrightarrow 0 < x \vee x < y$
	$x+y < y \leftrightarrow 0 < x \wedge 0 < y$	$0 < x \vee 0 < y \leftrightarrow 0 < x \vee x < y$
	$0 < x \wedge 0 < x+y \leftrightarrow y < x$	$0 < x \vee 0 < x+y \leftrightarrow 0 < x \vee x < y$
	$y < x \wedge (0 < x \vee x < y) \leftrightarrow y < x$	$0 < y \vee 0 < x+y \leftrightarrow 0 < x \vee x < y$
	$y < x+y \leftrightarrow y < x$	$0 < y \vee y < x \leftrightarrow 0 < x \vee x < y$

Fig. 5. Axiom Set in Exm. 16

Any universally quantified formula in the variables x, y and with \wedge and \vee as the only logical junctors that holds in $(\mathbb{N} \bmod 2)$ follows from the set of formulas given in Fig. 5; formulas with other quantifier prefixes are omitted.

Pure ground formulas and formulas that are instances of others have been manually deleted, as well as formulas that follow from propositional tautologies or symmetry of equality. Equations in *Nat* are listed first, with index $_{Nat}$ omitted, followed by equations in *Bool*, $=_{Bool}$ written as \leftrightarrow . Note that $(=_{Nat})$ is *not* contained in the signature of this example but was introduced by our method; consequently, no equality predicate appears in the equations of sort *Bool*, e.g. in a law like $x < y \vee x = y \vee y < x \leftrightarrow \text{true}$.

No formula at all can be reduced to *true* by the axioms in Fig. 5, indicating that no valid statements about $(\mathbb{N} \bmod 2)$ can be expressed in Σ , except for trivial propositional instances like $\text{true} \vee x < y$. In fact, every expressible formula (i.e. term in $\mathcal{T}_{\{x,y\}, Bool}(\Sigma)$) can be falsified by instantiating both x and y to 0. \square

In Cor. 17, we give an application in the field of automated theorem proving. Here, it is common practice to test a conjecture φ in a finite class \mathbf{A} of models of the background theory Θ before attempting to prove φ from Θ . If the test fails, it is clear that $\Theta \models \varphi$ cannot hold. If the test succeeds, i.e. $\mathbf{A} \text{ sat } \varphi$, we are usually still faced with the task of proving $\Theta \models \varphi$.

We call \mathbf{A} a class of prototype algebras for Θ , if from a succeeding test we always can conclude the validity of $\Theta \models \varphi$. In this case, we can decide quickly whether Θ entails a formula φ , merely by testing whether φ holds in each member of \mathbf{A} .

Corollary 17 provides a sufficient criterion for establishing the existence of prototype algebras for an equational – or by the above argument – first-order predicate-logic theory Θ . Since we cannot deal with arbitrarily many variables, we have to restrict the syntactic class of φ to $\mathcal{T}_{\mathbf{x}}(\Sigma \cup \{ (=_{Bool}) \})$ for some finite tuple \mathbf{x} of variables.

Example 18 gives a set of prototype algebras for an equational theory, and at the same time shows that they do not exist for arbitrary theories. It remains

to be seen whether it is feasible to extend the prototype approach by adding certain infinite algebras that allow easy testing⁶ of φ to \mathbf{A} .

Corollary 17. *[Prototype algebras] Let Θ be a set of formulas, let \mathbf{A} be a finite class of finite admitted Σ -algebras such that $\mathbf{A} \text{ sat } \Theta$ and $\Theta \models \mathcal{AX}_{\mathbf{x}}(\mathbf{A})$. Then, \mathbf{A} is a class of prototype algebras for Θ and \mathbf{x} . Formally, for any formula φ over Σ and \mathbf{x} we have $\Theta \models \varphi$ iff $\mathbf{A} \text{ sat } \varphi$.*

Proof.

“ \Rightarrow ”: trivial: $\mathbf{A} \text{ sat } \Theta \models \varphi$

“ \Leftarrow ”: $\mathbf{A} \text{ sat } \varphi$
 $\Rightarrow \mathcal{AX}_V(\mathbf{A}) \models \varphi$ by Thm. 13
 $\Rightarrow \Theta \models \varphi$ since $\Theta \models \mathcal{AX}_V(\mathbf{A})$

Note that the Corollary holds for arbitrary \mathcal{S}_{fix} and \mathcal{A}_{fix} , not only for *Bool*. However, we need to fix *Bool* in order to get the notion of \models used in theorem proving. \square

Example 18. Let Θ consist of the axioms for an Abelian group of characteristic 2. Referring to Exm. 14, we can see that Θ implies $\mathcal{AX}_{\langle x, y \rangle}(\mathcal{A}_2)$. Hence, in order to prove a formula φ over $\{x, y, 0, (+), (=)\}$ to be a consequence of Θ , it is sufficient by Cor. 17 just to test it in \mathcal{A}_2 .

Unfortunately, we cannot get rid of “characteristic” equations: In any finite algebra with an associative binary operation $(+)$, a law $n_1x = n_2x$ holds for some $n_1 > n_2$, where we abbreviate $nx := x + \dots + x$ (n times). Hence, each axiom set obtained from finitely many of such algebras necessarily entails a law of this form. \square

Acknowledgments. Ingo Dahn drew our attention to the application area of prototype algebras in automated theorem proving. Martin Simons provided the literature reference to the many-sorted version of Birkhoff’s variety theorem. Angela Sodan gave us some valuable advice on the presentation.

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⁶ In an equational setting, the singleton class containing the initial algebra is always a prototype. However, equality in the initial algebra is generally undecidable.

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